PROJECTIVE MULTI-RESOLUTION ANALYSES FOR $L^2(\mathbb{R}^2)$

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Abstract. We define the notion of "projective" multiresolution analyses, for which, by definition, the initial space corresponds to a finitely generated projective module over the algebra $C(\mathbb{T}^n)$ of continuous complex-valued functions on an n-torus. The case of ordinary multi-wavelets is that in which the projective module is actually free. We discuss the properties of projective multiresolution analyses, including the frames which they provide for $L^2(\mathbb{R}^n)$. Then we show how to construct examples for the case of any diagonal 2×2 dilation matrix with integer entries, with initial module specified to be any fixed finitely generated projective $C(\mathbb{T}^2)$-module. We compute the isomorphism classes of the corresponding wavelet modules.

In classical wavelet theory one uses multi-resolution analyses to construct (multi-) wavelets and their corresponding orthonormal bases or frames for $L^2(\mathbb{R}^n)$. In almost all applications the scaling functions and wavelets have continuous Fourier transforms. This continuity is a significant and interesting condition. If one requires just a bit more, then one finds that in the frequency domain one is dealing with what are called projective modules over $C(\mathbb{T}^n)$ (or equivalently, with the spaces of continuous cross-sections of complex vector bundles over $\mathbb{T}^n$). The case of a single scaling function or wavelet corresponds to the free module of rank 1, whereas the case of several orthogonal scaling functions or wavelets corresponds to free modules of higher rank. This leads one to ask whether wavelet theory carries over to the case of general projective modules over $C(\mathbb{T}^n)$. It is the purpose of this paper to show that the answer is affirmative.

For a given dilation matrix A we define a projective multiresolution analysis for $L^2(\mathbb{R}^n)$ to be an increasing sequence $\{V_j\}$ of subspaces having the usual properties, with the one exception that instead of V_0 being the linear span of the integer translates of one or more scaling functions, we only require that V_0 be the (projective) module of inverse Fourier transforms of continuous cross-sections of a complex vector bundle over $\mathbb{T}^n$. The precise definition is given in Definition 3.1. We show in Section 3 that projective multiresolution analyses give in a natural way wavelets which determine normalized tight frames for $L^2(\mathbb{R}^n)$. This seems to be a somewhat new way of constructing tight frames for $L^2(\mathbb{R}^n)$. We note that the term "projective multiresolution analysis" was first used by G. Zimmermann his thesis [17]; the definition given by him is different from the one used here, and we comment on the difference in Section 3.

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It is not immediately evident that non-trivial projective multiresolution analyses exist. We show how to construct them for any diagonal 2×2 dilation matrix and the continuous cross-section space of any complex vector bundle over $\mathbb{T}^2$. This fleshes out ideas which the second author presented in a special session talk at the annual meeting of the AMS in 1997 [15]. We also determine the isomorphism class of the corresponding wavelet modules. We find that in some cases the wavelet modules are free modules, while in other cases they are not free.

As hinted above, we find it very convenient to take a fairly algebraic approach to our topic. The function spaces determined by scaling functions and wavelets are viewed as modules over the convolution algebras of the corresponding discrete subgroups of $\mathbb{R}^n$. In the frequency domain these convolution algebras are (when suitably completed) isomorphic to $C(\mathbb{T}^n)$, which is a commutative C^*-algebra. As seen already in [10], it is very useful to define on modules over $C(\mathbb{T}^n)$ inner products with values in this algebra. Modules equipped with algebra-valued inner products have been used very profitably for three decades in related areas of non-commutative harmonic analysis [12] [8], and are often called "Hilbert C^*-modules". Recently there has been much interest in frames for Hilbert C^*-modules. Besides [10] see [4] [5] for discussions and references. Such frames turn out to be very useful in our context, leading naturally to ordinary tight frames for $L^2(\mathbb{R}^n)$.

Our paper is organized as follows. In Section 1 we discuss the setting for our topic, and describe in detail the standard $C(\mathbb{T}^n)$-module Ξ where we will perform most of our constructions. Roughly speaking, Ξ is the closure of $C_c(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ for a norm stronger than the Hilbert-space norm. In Section 2 we recall the definition and basic properties of projective modules, and of the frames which are associated with them. In Section 3 we define the notion of projective multiresolution analyses, and obtain many of their general properties. In particular, we discuss the module frames and the tight frames for $L^2(\mathbb{R}^n)$ associated with them. In Section 4 we briefly review specific constructions of certain finitely generated projective $C(\mathbb{T}^2)$ modules along the lines discusses in [14]. They provide model modules for each isomorphism class of projective modules over $C(\mathbb{T}^2)$. We mention some of their properties, and their relationship to the K-theory of $\mathbb{T}^2$.

In Section 5 we restrict attention to diagonal 2×2 dilation matrices. We present there one of our main technical results, namely, the construction of an appropriate substitute for a scaling function in Ξ corresponding to a desired initial projective module. We give a sufficient condition which guarantees that an initial $C(\mathbb{T}^2)$-submodule of Ξ generated by a single element of Ξ will gives rise to a projective multi-resolution analysis corresponding to a given dilation operator. We use this to show how to construct, for a given arbitrary projective $C(\mathbb{T}^2)$-module and an arbitrary diagonal dilation matrix, a projective multi-resolution analysis with its initial module V_0 isomorphic to the given projective module. In Section 6 we identify the structure and isomorphism classes of the wavelet modules corresponding to the projective multiresolution analyses constructed in the Section 5.

We should note that for diagonal matrices, some our constructions extend to higher dimensions, and we intend to publish these results in a later paper. For matrices which are not similar via an element of $GL(2, \mathbb{Z})$ to a diagonal matrix, the situation seems

more difficult; however we note that we have been able to constructed a projective multi-resolution analysis for the matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ whose initial module is not free. The methods used in our construction at this point seem somewhat ad-hoc; but one intriguing result from the calculation in this case is that the higher dimensional modules W_i are all free $C(\mathbb{T}^2)$-modules of dimension $2^{i+1} - 2^i$, which we conjecture corresponds to the determinant of A being positive, as in the diagonal case.

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1. The setting

In this section we describe the setting within which our construction of projective multiresolution analyses will take place. Much of the material in the first part of this section is well-known, but it serves to establish our notation. We first sketch our setting in the "time" domain, but then we indicate more precisely what the setting looks like in the "frequency" domain, where, as usual in this subject, matters are more transparent, and where we will work for the rest of the paper.

We will denote the algebra of bounded operators on $L^2(\mathbb{R}^n)$ by $\mathcal{L}(L^2(\mathbb{R}^n))$. For each $s \in \mathbb{R}^n$ we define the translation operator T_s on $L^2(\mathbb{R}^n)$ by $(T_s\xi)(t) = \xi(t-s)$. We denote by $\mathcal{A}$ the operator-norm closed subalgebra of the $\mathcal{L}(L^2(\mathbb{R}^n))$ generated by all the translation operators T_γ for $\gamma \in \mathbb{Z}^n$. Equivalently, $\mathcal{A}$ is generated by all the convolution operators T_f for $f \in \ell^1(\mathbb{Z}^n)$. We will see below that the elements of $\mathcal{A}$ will in our setting act somewhat like scalars, and that as is often done in related situations [8] [10] it is then convenient to use right-module notation, while putting other operators on the left. Thus for $f \in L^1(\mathbb{Z}^n)$ and $\xi \in L^2(\mathbb{R}^n)$ we have

$$(T_f\xi)(t) = (\xi * f)(t) = \sum_{\gamma\in\mathbb{Z}^n} f(\gamma)(T_\gamma\xi)(t) = \sum_{\gamma\in\mathbb{Z}^n} \xi(t-\gamma)f(\gamma).$$

We will denote by $\|f\|_{\mathcal{A}}$ the operator norm of T_f. The natural involution coming from taking adjoints of operators is defined by $(f^*)(\gamma) = \bar{f}(-\gamma)$.

Throughout we let A be an $n \times n$ dilation matrix, that is, an invertible matrix with integer entries, all of whose eigenvalues have modulus strictly greater than 1. Note that $A\mathbb{Z}^n \subset \mathbb{Z}^n$. Throughout we let $\delta = |\det(A)|^{-1/2}$. We let D denote the unitary dilation operator on $L^2(\mathbb{R}^n)$ defined by

$$(D\xi)(t) = \delta^{-1}\xi(At).$$

We let α denote the automorphism of $\mathcal{L}(L^2(\mathbb{R}^n))$ consisting of conjugating by D, that is, $\alpha(T) = DTD^{-1}$ for $T \in \mathcal{L}(L^2(\mathbb{R}^n))$. Now α does not carry $\mathcal{A}$ into itself, but rather onto the algebra of convolution operators determined by $\ell^1(A^{-1}\mathbb{Z}^n)$. To be specific,

$$(\alpha(f))(\gamma) = f(A\gamma)$$

for $f \in \ell^1(\mathbb{Z}^n)$ and $\gamma \in A^{-1}\mathbb{Z}^n \subseteq \mathbb{R}^n$. More generally, for any $j \in \mathbb{Z}$ let $\mathcal{A}_j$ denote the algebra of operators corresponding to $\ell^1(A^{-j}\mathbb{Z}^n)$. Then $\mathcal{A}_j \subset \mathcal{A}_{j+1}$ and $\alpha(\mathcal{A}_j) = \mathcal{A}_{j+1}$,

while $\mathcal{A}_j = \alpha^j(\mathcal{A})$, for each j. These algebras are all isomorphic, but we must be careful to distinguish between them because they are quite different sets of operators on $L^2(\mathbb{R}^n)$.

As mentioned in the introduction, in order for our topic to have content it is essential that we work with functions whose Fourier transforms are continuous. For this purpose it is natural to introduce an appropriate dense subspace of $L^2(\mathbb{R}^n)$, on which is defined an $\mathcal{A}$-valued "inner product". This inner product is defined by convolution followed by "down-sampling", namely

$$\langle \xi, \eta \rangle_{\mathcal{A}}(\gamma) = (\xi^* * \eta)(\gamma) = \int_{\mathbb{R}^n} \bar{\xi}(t)\eta(t+\gamma)\ dt.$$

Notice that this inner product is very natural within the context of wavelet theory (as seen also in [10]), since if $\varphi_1, \ldots, \varphi_m$ is a possible family of multi-scaling functions, or multi-wavelets, then the condition that all their translates by elements of $\mathbb{Z}^n$ form an orthonormal set is just the condition that

$$\langle \varphi_j, \varphi_k \rangle_{\mathcal{A}} = \delta_{jk} 1_{\mathcal{A}}$$

for all j, k, where $1_{\mathcal{A}}$ is the "delta-function" at 0 in $\ell^1(\mathbb{Z}^n)$. When we pass to the "frequency domain" it will be much clearer what the domain of this inner product should be. But the Schwartz functions will be dense in the domain, and so the reader can temporarily assume that the domain consists of the Schwartz functions. It is for this inner product that the elements of $\mathcal{A}$ act much like scalars, in the sense that

$$\langle \xi, \eta * f \rangle_{\mathcal{A}} = \langle \xi, \eta \rangle_{\mathcal{A}} * f.$$

In terms of this inner product we can define an ordinary norm in the same way as one does for Hilbert spaces, namely

$$\|\xi\|_{\mathcal{A}} = (\|\langle \xi, \xi \rangle_{\mathcal{A}}\|_{\mathcal{A}})^{1/2}.$$

We can define in a similar way inner products on Ξ with values in each $\mathcal{A}_j$. Then we will have

$$\langle D\xi, D\eta \rangle_{\alpha(\mathcal{A}_j)} = \alpha(\langle \xi, \eta \rangle_{\mathcal{A}_j}),$$

and similarly for powers of D. Since α is an isometry for the operator norm, it follows that D is isometric, in the sense that $\|D\xi\|_{\alpha(\mathcal{A}_j)} = \|\xi\|_{\mathcal{A}_j}$. It is also easy to check that

$$D(\xi * f) = (D\xi)(\alpha(f)).$$

We now pass to the frequency domain. We define on $\mathbb{R}$ the function e by $e(r) = e^{2\pi i r}$, and we define the Fourier transform, $\mathcal{F}$, by

$$\mathcal{F}(\xi)(x) = \int_{\mathbb{R}^n} \xi(t)\bar{e}(t \cdot x)\ dt.$$

For the rest of the paper we will work in the frequency domain, and so for notational simplicity we will make the unusual choice of denoting the Fourier transforms of objects by the same notation as the original objects. Usually only the variables will indicate that we are in the frequency domain.

Simple calculations show what happens to the setting described above under the Fourier transform. We record here some of the facts, many of them familiar. The algebra $\mathcal{A} = \mathcal{A}_0$ becomes the algebra $\mathcal{A} = C(\mathbb{T}^n)$ of continuous functions on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The action of $\mathcal{A}$

on $L^2(\mathbb{R}^n)$ is now given simply by pointwise multiplication, where we view the functions in $\mathcal{A}$ as periodic functions on $\mathbb{R}^n$. That is,

$$(\xi f)(x) = \xi(x)f(x).$$

The operator norm on $\mathcal{A}$ is now just the usual supremum norm $\|\cdot\|_\infty$, and the involution is complex conjugation. We have

$$(\mathcal{F}D\mathcal{F}^*)(x) = \delta\xi((A^t)^{-1}x),$$

where A^t is the transpose of A. Set $B = (A^t)^{-1}$. Then we see that in the frequency domain we deal with the operator $\mathcal{F}D\mathcal{F}^*$. As we will primarily be working in the frequency domain, there is no danger of confusion by denoting this operator by D as well, and D is given by

$$(D\xi)(x) = \delta\xi(Bx).$$

To see what the $\mathcal{A}$-valued inner product will be, we first determine the Fourier transform of the down-sampling operator, which we now denote by S. Let ξ be a Schwartz function in the time domain. Define e_x by $e_x(t) = e(t \cdot x)$. By the Poisson summation formula we obtain for $x \in \mathbb{R}^n$

$$\begin{aligned}(S\xi)\check{}(x) &= \sum_\gamma \xi(\gamma)\bar{e}(\gamma \cdot x) = \sum_p (\xi\bar{e}_x)\check{}(p) \\ &= \sum_p \int (\xi\bar{e}_x)(t)\bar{e}(t\cdot p)\,dt = \sum_p \int \xi(t)\bar{e}(t\cdot x)\bar{e}(t\cdot p)\,dt = \sum_p \hat{\xi}(x+p).\end{aligned}$$

Thus for η a Schwartz function in the frequency domain we have

$$(S\eta)(x) = \sum_p \eta(x+p) = \sum_p \eta(x-p).$$

In view of this, we see that the inner product in the frequency domain is given by

$$\langle \xi, \eta\rangle_{\mathcal{A}}(x) = \sum_{p\in\mathbb{Z}^n} (\bar{\xi}\eta)(x-p).$$

It is clear from this that

$$\langle \xi, \eta f\rangle_{\mathcal{A}} = \langle \xi, \eta\rangle_{\mathcal{A}} f$$

for $f \in \mathcal{A}$. We can now see more easily what the domain, Ξ, for this inner product should be.

Various versions of the space Ξ below have been introduced and studied before, specifically by G. Zimmermann in Chapter V of his thesis [17], where the notation $L^{2,\infty}(\mathbb{Z}^n, \mathbb{T}^n)$ was used, again by Zimmermann in [18], and by J. Benedetto and Zimmermann in [2]. We thank the referee for bringing these works to our attention. We use different notation here, which we specify in the following definition.

Definition 1.1. We let Ξ denote the set of bounded continuous functions on $\mathbb{R}^n$ for which there is a constant, K, such that $\sum_{p\in\mathbb{Z}^n} |\xi(x-p)|^2 \le K$ for each $x \in \mathbb{R}^n$, and furthermore such that the function defined by this sum is continuous. The norm is given by

$$\|\xi\|_{\mathcal{A}} = \|\langle\xi,\xi\rangle_{\mathcal{A}}\|^{1/2} = \sup_x (\sum_{p\in\mathbb{Z}^n} |\xi(x-p)|^2)^{1/2}.$$

We remark that when one sees the naturalness of the proofs of Propositions 3.2 and 3.4 one understands that Ξ with its $\mathcal{A}$-valued inner product is a quite comfortable setting within which to develop even ordinary wavelet theory.

Proposition 1.2. *For any $\xi, \eta \in \Xi$ the sum defining $\langle \xi, \eta \rangle_{\mathcal{A}}$ converges uniformly on compact subsets of $\mathbb{R}^n$, and we have*

$$\langle \xi, \eta \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{T}^n} \langle \xi, \eta \rangle_{\mathcal{A}} \quad .$$

For any $\xi \in \Xi$ we have

$$\|\xi\|_2 \leq \|\xi\|_{\mathcal{A}} \qquad \text{and} \qquad \|\xi\|_\infty \leq \|\xi\|_{\mathcal{A}}.$$

Proof. Dini's lemma (page 34 of [11]) tells us that the sum for $\langle \xi, \xi \rangle_{\mathcal{A}}$ will converge uniformly on compact subsets of $\mathbb{R}^n$. The corresponding fact for $\langle \xi, \eta \rangle_{\mathcal{A}}$ then follows by polarization. For any $\xi, \eta \in \Xi$ we have

$$\langle \xi, \eta \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \bar{\xi}(x)\eta(x)\ dx = \int_{\mathbb{T}^n} (\sum_{p \in \mathbb{Z}^n} \bar{\xi}(x-p)\eta(x-p))\ dx = \int_{\mathbb{T}^n} \langle \xi, \eta \rangle_{\mathcal{A}}(x)\ dx.$$

The first inequality follows immediately from this. The second inequality is clear from the definitions. $\square$

We will need the following version of the Cauchy-Schwarz inequality for our inner product. (See proposition 2.9 of [12], and [8].)

Proposition 1.3. *For any $\xi, \eta \in \Xi$ we have*

$$|\langle \xi, \eta \rangle_{\mathcal{A}}|^2 \leq \langle \xi, \xi \rangle_{\mathcal{A}} \langle \eta, \eta \rangle_{\mathcal{A}}$$

as an inequality between functions in $\mathcal{A}$.

Proof. For any $\zeta \in \Xi$ and any $x \in \mathbb{R}^n$ define $\zeta^x \in \ell^2(\mathbb{Z}^n)$ by $\zeta^x(p) = \zeta(x-p)$. Then for $\xi, \eta \in \Xi$ we have, upon using the ordinary Cauchy-Schwarz inequality,

$$\begin{aligned} |\langle \xi, \eta \rangle_{\mathcal{A}}(x)|^2 &= |\sum_p \bar{\xi}^x(p)\eta^x(p)|^2 \\ &\leq (\sum_p |\xi^x(p)|^2)(\sum_q |\eta^x(q)|^2) = \langle \xi, \xi \rangle_{\mathcal{A}}(x) \langle \eta, \eta \rangle_{\mathcal{A}}(x). \end{aligned}$$

$\square$

Standard arguments of the kind used to treat ℓ^2 show that Ξ is a linear space on which the values of the inner product are in $\mathcal{A}$. We now introduce common notation which we will use here and in a number of places later. We let $C_b(\mathbb{R}^n)$ denote the algebra of continuous bounded functions on $\mathbb{R}^n$, we let $C_\infty(\mathbb{R}^n)$ denote its subalgebra of functions which vanish at infinity, and we let $C_c(\mathbb{R}^n)$ denote its subalgebra of functions of compact support. We let I^n denote the cube $[0,1]^n$. Every coset of $\mathbb{Z}^n$ in $\mathbb{R}^n$ meets I^n. Thus sums such as that defining $\langle \xi, \eta \rangle_{\mathcal{A}}(x)$ need only be considered for $x \in I^n$.

Proposition 1.4. *The space Ξ is complete for its norm. Furthermore, $\Xi \subset C_\infty(\mathbb{R}^n)$.*

Proof. Suppose that $\{\xi_k\}$ is a Cauchy sequence in Ξ. Then from Proposition 1.2 it follows that this sequence is Cauchy for $\|\cdot\|_\infty$, and so converges uniformly to some $\xi \in C_b(\mathbb{R}^n)$. For any $\eta \in \Xi$ and any $x \in \mathbb{R}^n$ let η^x be as in the proof or Proposition 1.3. Because $\{\xi_k\}$ is a Cauchy sequence, so is $\{\xi_k^x\}$ for each $x \in \mathbb{R}^n$, and $\{\xi_k^x\}$ will converge in $\ell^2(\mathbb{Z}^n)$, necessarily to ξ^x. Furthermore, from the definition of $\|\cdot\|_{\mathcal{A}}$ this convergence is uniform in x. It follows that $\|\xi\|_{\mathcal{A}} = \sup_x \|\xi^x\|_2$, so that $\xi \in \Xi$. A bit of further argument shows that for any $\varepsilon > 0$ there is a k_0 such that if $k \geq k_0$ then $\|\xi^x - \xi_k^x\|_2 < \varepsilon$ for all $x \in \mathbb{R}^n$, so that $\{\xi_k\}$ converges to ξ for the norm of Ξ.

Finally, let $\xi \in \Xi$ be given. For any given $\varepsilon > 0$ we can find a finite subset S of $\mathbb{Z}^n$ such that $\sum_{p\notin S} |\xi(x-p)|^2 < \varepsilon$ for all $x \in I^n$. Then, in particular, $|\xi(x-p)|^2 < \varepsilon$ for every $p \notin S$ and every $x \in I^n$. Thus if $|\xi(y)|^2 > \varepsilon$ for some $y \in \mathbb{R}^n$, we have $y \in I^n - S$ (the union of translates of I^n by elements of S), which is compact. Consequently, $\xi \in C_\infty(\mathbb{R}^n)$. □

Proposition 1.5. *If $F \in C_b(\mathbb{R}^n)$ then the pointwise product ξF is in Ξ for each $\xi \in \Xi$, and $\|\xi F\|_{\mathcal{A}} \leq \|\xi\|_{\mathcal{A}} \|F\|_\infty$. Furthermore, $C_c(\mathbb{R}^n)$ is dense in Ξ for its norm.*

Proof. For any finite subset S of $\mathbb{Z}^n$ we have

$$\sum_S |(\xi F)(x-p)|^2 \leq \|F\|_\infty \sum_S |\xi(x-p)|^2,$$

and from this it easily follows that $\xi F \in \Xi$ and $\|\xi F\|_{\mathcal{A}} \leq \|\xi\|_{\mathcal{A}} \|F\|_\infty$.

It is easily seen that $C_c(\mathbb{R}^n) \subset \Xi$. We now show that $C_c(\mathbb{R}^n)$ is dense. Given ξ and $\varepsilon > 0$, choose a finite set $S \subset \mathbb{Z}^n$ such that $\sum_{p\notin S} |\xi(x-p)|^2 < \varepsilon$ for all $x \in I^n$. Choose an $F \in C_b(\mathbb{R}^n)$ such that $F = 0$ on $I^n - S$ and $F(y) = 1$ for all y outside some compact neighborhood of $I^n - S$. Then $\xi(1-F) \in C_c(\mathbb{R}^n)$ and $\|\xi - \xi(1-F)\|_{\mathcal{A}} = \|\xi F\|_{\mathcal{A}} < \varepsilon$. □

Proposition 1.6. *Let $\xi \in C_b(\mathbb{R}^n)$. If there is a constant, K, and an $s > n/2$ such that $|\xi(x)| \leq K\|x\|^{-s}$ for all sufficiently large x, then $\xi \in \Xi$.*

Proof. For $x \in I^n$ and all sufficiently large $p \in \mathbb{Z}^n$ we will have $|\xi(x-p)| \leq 2K\|p\|^{-s}$. Since $\sum_p \|p\|^{-2s} < \infty$, it follows that $\sum |\xi(x-p)|^2$ converges uniformly for $x \in I^n$, to a necessarily continuous function. By translating by $\mathbb{Z}^n$ it follows that the sum converges uniformly on any compact subset of $\mathbb{R}^n$, to a continuous function. □

Corollary 1.7. *The space Ξ contains all the Schwartz functions on $\mathbb{R}^n$.*

Proposition 1.8. *The action of $\mathbb{R}^n$ on Ξ by translation is isometric and strongly continuous.*

Proof. That the action is isometric is evident from the definition of the norm. Since the action is isometric, it suffices to show that the action is strongly continuous on $C_c(\mathbb{R}^n)$, since $C_c(\mathbb{R}^n)$ is dense by Proposition 1.5. Now if $\xi \in C_c(\mathbb{R}^n)$ then for z's and x's restricted to given compact sets there is only a finite number of p's in $\mathbb{Z}^n$ such that $(T_z\xi)(x-p) - \xi(x-p) \neq 0$. The strong continuity then follow from the fact that ξ is uniformly continuous since it is in $C_c(\mathbb{R}^n)$. □

Corollary 1.9. *The infinitely differentiable functions of compact support are dense in Ξ.*

Proof. Since translation is strongly continuous, the usual smoothing argument works, consisting of convolving functions in $C_c(\mathbb{R}^n)$ by smooth functions of compact support. □

By definition [8], a "right Hilbert C^*-module" over a C^*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module which is equipped with an $\mathcal{A}$-valued inner product and which is complete for the corresponding norm (defined much as in Definition 1.1). In terms of this definition we can summarize the early part of the above discussion by:

Proposition 1.10. *When Ξ is equipped with its $\mathcal{A}$-valued inner product and right action of $\mathcal{A}$, it is a right Hilbert C^*-module over $\mathcal{A}$ (or a "right Hilbert $\mathcal{A}$-module").*

We remark that many of the considerations above apply in the setting of an arbitrary locally compact group G (or a Lie group) and a discrete subgroup G_0 such that G/G_0 is compact.

For each j we find that $\mathcal{A}_j = C(\mathbb{R}^n/B^{-j}\mathbb{Z}^n)$. Since $B^{-j}\mathbb{Z}^n \supset B^{-(j+1)}\mathbb{Z}^n$, we have $\mathcal{A}_j \subset \mathcal{A}_{j+1}$ as expected, where again we view elements of $C(\mathbb{R}^n/B^{-j}\mathbb{Z}^n)$ as functions on $\mathbb{R}^n$. We also have

$$(\alpha(f))(x) = f(Bx)$$

for $f \in \mathcal{A}_j$. For each algebra $\mathcal{A}_j$ we have a corresponding $\mathcal{A}_j$-valued inner product on a dense subspace of $L^2(\mathbb{R}^n)$. But in fact these dense subspaces all coincide with Ξ, because the corresponding norms are all equivalent. In fact we have:

Proposition 1.11. *For any given $j \in \mathbb{Z}$ let Γ be the image of $B^{-j}\mathbb{Z}^n$ in $\mathbb{R}^n/(B^{-(j+1)}\mathbb{Z}^n)$, so that Γ is a finite group. Then for any $\xi, \eta \in \Xi$ we have*

$$\langle \xi, \eta \rangle_{\mathcal{A}_j}(x) = \sum_{s \in \Gamma} \langle \xi, \eta \rangle_{\mathcal{A}_{j+1}}(x - s)$$

for all $x \in \mathbb{R}^n$. We also have

$$\|\xi\|_{\mathcal{A}_{j+1}} \leq \|\xi\|_{\mathcal{A}_j} \leq \delta^{-1}\|\xi\|_{\mathcal{A}_{j+1}}.$$

Proof. For simplicity of notation we just compare $\mathcal{A} = \mathcal{A}_0$ with $\mathcal{A}_1$, so that $j = 0$. Pull back Γ to a set C of coset representatives for $B^{-1}\mathbb{Z}^n$ in $\mathbb{Z}^n$, so that

$$|C| = |\det(B^{-1})| = |\det(A)| = \delta^{-2},$$

where $|C|$ denotes the number of elements in C. Then for any $x \in \mathbb{R}^n$ we have

$$\langle \xi, \eta \rangle_{\mathcal{A}}(x) = \sum_{\mathbb{Z}^n} (\bar{\xi}\eta)(x - p) = \sum_{c \in C} \sum_{q \in \mathbb{Z}^n} (\bar{\xi}\eta)(x - c - B^{-1}q) = \sum_{s \in \Gamma} \langle \xi, \eta \rangle_{\mathcal{A}_1}(x - s).$$

The second inequality follows easily from this. The first inequality follows immediately from $B^{-1}\mathbb{Z}^n \subset \mathbb{Z}^n$. $\square$

Corollary 1.12. *Let $\xi, \eta \in \Xi$. If $\langle \xi, \eta \rangle_{\mathcal{A}_j} = 0$ for some j, then $\langle \xi, \eta \rangle_{\mathcal{A}_k} = 0$ for all $k < j$.*

We point out the following view of the above discussion. It is natural to define a "conditional expectation", E, from $\mathcal{A}_1$ onto $\mathcal{A}$ by averaging over Γ. That is,

$$E(g)(x) = \delta^2 \sum_{s \in \Gamma} g(x - s)$$

for $g \in \mathcal{A}_1$. It is clear that $\|E(g)\|_\infty \leq \|g\|_\infty$. Then, the first part of the above proposition says that

$$\langle \xi, \eta \rangle_{\mathcal{A}_j} = \delta^{-2} E(\langle \xi, \eta \rangle_{\mathcal{A}_{j+1}}),$$

while the second inequality of the above proposition amounts to

$$\|\langle\xi,\xi\rangle_{\mathcal{A}_j}\|_\infty = \delta^{-2}\|E(\langle\xi,\xi\rangle_{\mathcal{A}_{j+1}})\|_\infty \le \delta^{-2}\|\langle\xi,\xi\rangle_{\mathcal{A}_{j+1}}\|_\infty.$$

The following formulas, whose versions in the time domain were mentioned earlier, are verified by straightforward calculations.

Proposition 1.13. *For any $\xi,\eta\in\Xi$ and $f\in\mathcal{A}_j$ we have*

$$D(\xi f) = (D\xi)(\alpha(f)) \qquad \text{and} \qquad \langle D\xi, D\eta\rangle_{\mathcal{A}_{j+1}} = \alpha(\langle\xi,\eta\rangle_{\mathcal{A}_j}).$$

2. Projective modules and frames

Suppose now that $\varphi_1,\ldots,\varphi_m\in\Xi$ and that $\langle\varphi_i,\varphi_j\rangle_{\mathcal{A}} = \delta_{ij}1_{\mathcal{A}}$. Then it is easily checked that the transformation

$$(a_1,\ldots,a_m)\mapsto\sum^m\varphi_j a_j$$

from $\mathcal{A}^m$ into Ξ is a right $\mathcal{A}$-module homomorphism which is isometric for the $\mathcal{A}$-valued inner products. Here $\mathcal{A}^m$ is viewed as a right $\mathcal{A}$-module in the evident way, and is equipped with the natural inner product

$$\langle(a_i),(b_j)\rangle_{\mathcal{A}} = \sum^m a_k^* b_k.$$

Then the $\mathcal{A}$-submodule of Ξ generated by $\varphi_1,\ldots,\varphi_m$ is norm-closed, and is isometrically isomorphic to $\mathcal{A}^m$. Now the $\mathcal{A}$-modules which are isomorphic to ones of the form $\mathcal{A}^m$ are exactly what are called (finitely generated) free $\mathcal{A}$-modules. We observe that in the time domain the $\mathcal{A}$-module generated by the φ_i's is exactly the closed span of the translates of the φ_i's by elements of $\mathbb{Z}^n$ (and these translates are all orthogonal for the ordinary inner product on $L^2(\mathbb{R}^n)$). This is the standard situation considered for wavelets and multi-wavelets.

The main thrust of our paper is that one can use somewhat more general $\mathcal{A}$-modules for the same purposes, namely (finitely generated) projective modules. Their usefulness in connection with wavelets was already demonstrated in [10]. For the reader's convenience we repeat here our brief discussion in [10] of the definition of projective modules, and of a few facts about them.

By definition, a projective module is (isomorphic to) a direct summand of a free module. In this paper all projective modules will be taken to be finitely generated. In the next section we will give specific examples of non-free projective $\mathcal{A}$-modules for the case $n=2$. Let us view our free module as $\mathcal{A}^m$ for some integer m, with "standard basis" $\{e_j\}$. Then the definition means that there is an $m\times m$ matrix P with entries in $\mathcal{A}$ which is a projection, that is $P^2=P$, such that our projective module V is of the form $V=P\mathcal{A}^m$. In this (C*-algebraic) setting a standard argument (see 5Bb in [16]) shows that P can be adjusted so that it is also "self-adjoint" in the evident sense. Let us then set $\xi_j = Pe_j$ for each j. For any $\eta\in V$ we have

$$\eta = P\eta = P(\sum e_j\langle e_j,\eta\rangle_{\mathcal{A}}) = \sum Pe_j\langle e_j,\eta\rangle_{\mathcal{A}} = \sum \xi_j\langle e_j,P\eta\rangle_{\mathcal{A}}$$

$$= \sum\xi_j\langle Pe_j,\eta\rangle_{\mathcal{A}} = \sum\xi_j\langle\xi_j,\eta\rangle_{\mathcal{A}}.$$

There is no reason to expect that the ξ_j's will be independent over $\mathcal{A}$, much less orthonormal. But anyone familiar with wavelets will feel comfortable about referring to the ξ_j's as a "module frame" for V. In the more general setting of projective modules over C*-algebras the above reconstruction formula

$$\eta \;=\; \sum \xi_j \; \langle \xi_j, \eta \rangle_{\mathcal{A}}$$

appears, in different notation, already in [13].

In the study of Hilbert modules the concept of module frames is extremely useful. We modify for our purposes some of the definitions and results about module frames given in the paper of Larson and Frank [4], starting with the following definition:

Definition 2.1. Let V be a finitely-generated right Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. By a *standard module frame* for V we mean a finite collection $\varphi_1, \dots, \varphi_m$ of elements of V for which the reconstruction formula

$$v = \sum_j \varphi_j \langle \varphi_j, v \rangle_{\mathcal{A}}$$

holds for all $v \in V$.

For use in Theorem 3.10 we also give the definition of standard module frames for Hilbert C^*-modules that are not finitely generated, due to Frank and Larson [4, 5].

Definition 2.2. Let Ξ be a right Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. We say that a countable subset $\{\varphi_j\}_{j \in I}$ of Ξ is a *standard module frame* for Ξ if for every $\xi \in \Xi$,

$$\langle \xi, \xi \rangle_{\mathcal{A}} \;=\; \sum_{j \in I} \langle \xi, \varphi_j \rangle_{\mathcal{A}} \langle \varphi_j, \xi \rangle_{\mathcal{A}},$$

where the sum on the right-hand side converges in norm in $\mathcal{A}$.

Frank and Larson have shown in Theorem 4.1 of [5] that if $\{\varphi_j\}_{j \in I}$ is a standard module frame for Ξ, then the reconstruction formula

$$\xi \;=\; \sum_{j \in I} \varphi_j \langle \varphi_j, \xi \rangle_{\mathcal{A}} \tag{1}$$

holds for every $\xi \in \Xi$, where the sum converges in norm. In the case where the index set I is finite, one can easily verify that the two definitions 2.1 and 2.2 of standard module frames are equivalent. For the moment we concentrate on Hilbert C^* submodules of Ξ having standard module frames of finite cardinality; examples of standard module frames of countably infinite cardinality will be given in Theorem 3.10 in the next section.

The discussion before the definitions shows that any (finitely generated) projective module has a standard module frame, in the sense of Definition 2.1. We now show, in two steps, that the converse is true.

Proposition 2.3. *Let V be a submodule of some Hilbert C^*-module Ξ over some unital C^*-algebra $\mathcal{A}$ which has a standard module frame $\{\varphi_j\}$ of finite cardinality. Then V is an $\mathcal{A}$-module direct summand of Ξ; a projection operator, P, from Ξ onto V is given by*

$$P(\xi) = \sum \varphi_j \langle \varphi_j, \xi \rangle_{\mathcal{A}}.$$

Proof. By Definition 2.1 of a standard module frame it is clear that the restriction of P to V is the identity operator on V. Thus $P^2 = P$. A simple calculation shows that $P^* = P$ in the sense that $\langle P\xi, \eta\rangle_{\mathcal{A}} = \langle \xi, P\eta\rangle_{\mathcal{A}}$. □

We remark that the above proposition is false, in general, for submodules V of Ξ having a countably infinite standard module frame $\{\varphi_j\}_{j=1}^{\infty}$. The problem arises because the sum $\sum \varphi_j\langle\varphi_j, \xi\rangle_{\mathcal{A}}$ need not converge in norm for every ξ in Ξ, although the sum will converge in norm for $\xi \in V$. We thank the referee for bringing this point to our attention.

Proposition 2.4. *Let V be a Hilbert C^*-module over $\mathcal{A}$ which has a standard module frame of finite cardinality. Then V can be embedded into some free module $\mathcal{A}^m$, with preservation of the $\mathcal{A}$-valued inner products. Furthermore, V is a projective $\mathcal{A}$ module.*

Proof. Let $\varphi_1, \dots, \varphi_m$ be a standard module frame for V. Define $T : V \mapsto \mathcal{A}^m$ by

$$Tv = (\langle\varphi_1, v\rangle_{\mathcal{A}}, \dots, \langle\varphi_m, v\rangle_{\mathcal{A}}).$$

Simple calculations show that T gives the desired embedding. The fact that V is projective then follows from Proposition 2.3. □

We now describe the relationship with ordinary frames for L^2-spaces within our specific setting of $\mathcal{A} = C(\mathbb{T}^n)$ and our $\Xi \subset L^2(\mathbb{R}^n)$. For $q \in \mathbb{Z}^n$ we let $e_q \in C(\mathbb{T}^n)$ be defined by $e_q(x) = \exp(2\pi i x \cdot q)$.

Proposition 2.5. *Let V be a projective $\mathcal{A}$-submodule of Ξ, and let $\{\varphi_k\}$ be a standard module frame for V. Then $\{\varphi_k e_q\}$ is a normalized tight frame for the closure of V in $L^2(\mathbb{R}^n)$.*

Proof. For any $\xi, \eta \in \Xi$ we have, by easily justified manipulations,

$$\begin{aligned}
\sum_{k,q} \overline{\langle\varphi_k e_q, \xi\rangle}_{L^2}\langle\varphi_k e_q, \eta\rangle_{L^2} &= \sum_{k,q}\int \overline{\langle\varphi_k e_q, \xi\rangle}_{\mathcal{A}}\int\langle\varphi_k e_q, \eta\rangle_{\mathcal{A}} \\
= \ \sum_k\Big(\sum_q \int e_q\overline{\langle\varphi_k, \xi\rangle}_{\mathcal{A}}\int \bar{e}_q\langle\varphi_k, \eta\rangle_{\mathcal{A}}\Big) &= \sum_k \int \overline{\langle\varphi_k, \xi\rangle}_{\mathcal{A}}\langle\varphi_k, \eta\rangle_{\mathcal{A}} \\
= \ \int \langle \sum_k \varphi_k\langle\varphi_k, \xi\rangle_{\mathcal{A}}, \eta\rangle_{\mathcal{A}} &= \int\langle\xi, \eta\rangle_{\mathcal{A}} = \langle\xi, \eta\rangle_{L^2}.
\end{aligned}$$

This relation then extends to the closure of V in $L^2(\mathbb{R}^n)$. Thus the basic condition in the definition of a normalized tight frame has been verified. A similar calculation shows that

$$\xi = \sum_{k,q}(\varphi_k e_q)\langle(\varphi_k e_q), \xi\rangle_{L^2}$$

for any $\xi \in V$, and so for any ξ in the closure of V in $L^2(\mathbb{R}^n)$. □

In the same way, standard module frames for projective submodules of Ξ over any $\mathcal{A}_j$ give normalized tight frames for their closures in $L^2(\mathbb{R}^n)$.

3. Projective multiresolution analyses

The main point of our paper is that it is possible to construct multiresolution analyses from projective modules which are not free. Explicit constructions of such multiresolution analyses will be given in the next sections. Here we just give the definition of such analyses, in terms of the notation which we have been using, and then explore some of their general properties.

Definition 3.1. By a *projective multiresolution analysis* for dilation by A we mean a family $\{V_j\}_{j\in\mathbb{Z}}$ of subspaces of Ξ such that

(1) V_0 is a projective $\mathcal{A}$-submodule of Ξ.
(2) $V_j = D^j(V_0)$ for all j.
(3) $V_j \supset V_{j-1}$ for all j.
(4) $\bigcup^\infty V_j$ is dense in Ξ
(5) $\bigcap_{-\infty}^\infty V_j = \{0\}$.

We comment on the relationship between our definition and the definition of projective multiresolution analyses given by Zimmermann in his thesis [17]. In Chapter VII of [17], a projective multiresolution analysis with scale $k \in \mathbb{N}\backslash\{1\}$ is defined to be a bisequence of continuous projections $\{P_m\}_{m\in\mathbb{Z}}$ defined on a translation-invariant locally convex topological vector space $X(\mathbb{R})$ of functions or distributions on the real line which satisfy:

[P1] $P_{m_1}P_{m_2} = P_{\min\{m_1,m_2\}}$;
[P2] For every $f \in X(\mathbb{R})$, $\lim_{m\to-\infty} P_m(f) = 0$ and $\lim_{m\to\infty} P_m(f) = f$;
[P3] For every $m \in \mathbb{Z}$, $D_kP_m = P_{m+1}D_k$, where D_k is the operator on $X(\mathbb{R})$ of dilation by k;
[P4] For all $m, n \in \mathbb{Z}$, $\tau_{k^{-m}n}P_m = P_m\tau_{k^{-m}n}$, where for $t \in \mathbb{R}$, τ_t is the operator of translation by t defined on $X(\mathbb{R})$, and $k \in \mathbb{N}\backslash\{1\}$ is the dilation factor;
[P5] There exists $\phi \in X(\mathbb{R})$, $\phi^* \in X(\mathbb{R})^*$ such that $\{\tau_n\phi, \tau_n\phi^*\}_{n\in\mathbb{Z}}$ is a biorthogonal system with $P_0(f) = \sum_{n\in\mathbb{Z}}\langle f, \tau_n\phi^*\rangle\tau_n\phi$.

We note that given a sequence of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of Ξ satisfying Definition 3.1 it is possible to form continuous projections $\{P_m\}$ on Ξ that satisfy conditions analogous to [P1]-[P4] of Zimmermann's definition with respect to the dilation A, first for the case $m \geq 0$ by using the constructions of projections given in Proposition 2.3, and then modifying this method for the case $m < 0$. However we have no analogue of [P5] in our definition, and our main aim is to construct C^*-module frames, as opposed to Zimmermann's emphasis on constructing an analogue of biorthogonal frames in his setting. We thank the referee for bringing Zimmermann's work to our attention.

Returning to Definition 3.1, we quickly point out that condition (5) above is actually redundant in our situation. That is, even more generally, assuming only (1) and (2) we obtain the following result:

Proposition 3.2. *Let V be any projective $\mathcal{A}$-submodule of Ξ, and set $V_j = D(V_{j-1})$ for all j. Then $\bigcap_{-\infty}^\infty V_j = \{0\}$.*

Proof. Let $\xi \in \bigcap_{-\infty}^{\infty} V_j$. Then $D^{-j}\xi \in V$ for each j. Let $\{\eta_k\}$ be a frame for V. Thus $D^{-j}\xi = \sum_k \eta_k \langle \eta_k, D^{-j}\xi\rangle_{\mathcal{A}}$, so that

$$\xi = \sum_k D^j(\eta_k \langle \eta_k, D^{-j}\xi\rangle_{\mathcal{A}})$$

for each j. We will try to show that $\langle g, \xi\rangle = 0$ for every $g \in C_c(\mathbb{R}^n)$. Now for such g we have

$$\langle g, \xi\rangle = \sum_k \langle g, D^j(\eta_k \langle \eta_k, D^{-j}\xi\rangle_{\mathcal{A}})\rangle.$$

So we examine the individual terms of this sum.

For fixed k let $F_j = \langle \eta_k, D^{-j}\xi\rangle_{\mathcal{A}}$. Then, using the Cauchy-Schwarz inequality of Proposition 1.3, and then Proposition 1.2, we have

$$\begin{aligned}
\|F_j\|_2^2 &= \int_{\mathbb{T}^n} |F_j|^2 = \int |\langle \eta_k, D^{-j}\xi\rangle_{\mathcal{A}}|^2 \\
&\leq \int \langle \eta_k, \eta_k\rangle_{\mathcal{A}} \langle D^{-j}\xi, D^{-j}\xi\rangle_{\mathcal{A}} \leq \|\langle \eta_k, \eta_k\rangle_{\mathcal{A}}\|_\infty \int \langle D^{-j}\xi, D^{-j}\xi\rangle_{\mathcal{A}} \\
&= \|\eta_k\|_{\mathcal{A}}^2 \|D^{-j}\xi\|_2^2 = \|\eta_k\|_{\mathcal{A}}^2 \|\xi\|_2^2.
\end{aligned}$$

Thus

$$\|F_j\|_2 \leq \|\eta_k\|_{\mathcal{A}} \|\xi\|_2.$$

Note that the right-hand side is independent of j.

Because A is a dilation, a bit of examination shows that we can hope that as $j \to -\infty$ the "mass" of $D^j(\eta_k\langle \eta_k, D^{-j}\xi\rangle_{\mathcal{A}})$ will converge towards $0 \in \mathbb{R}^n$. In order to quantify this idea, for any $r > 0$ we define the function χ_r on $\mathbb{R}^n$ by giving it value 1 outside the ball of radius r, and value 0 inside the ball.

Lemma 3.3. *Let $\eta \in \Xi$. For any $\varepsilon > 0$ there is an $r > 0$ such that for all $F \in \mathcal{A}$ we have*

$$\|\chi_r \eta F\|_2 \leq \varepsilon \|F\|_2.$$

Proof. The sum $\sum |\eta(x-p)|^2$ converges uniformly for $x \in I^n$, and so we can find a finite subset, S, of $\mathbb{Z}^n$ such that $\sum_{p \notin S} |\eta(x-p)|^2 < \varepsilon^2$ for all $x \in I^n$. If $p \in S$ and $x \in I^n$ then $\|x - p\| \leq \|x\| + \|p\| \leq \sqrt{n} + \|p\|$. Set

$$r = \sqrt{n} + \max\{\|p\| : p \in S\} + 1.$$

Thus if $\chi_r(x-p) \neq 0$ for an $x \in I^n$, then $\|x-p\| \geq r$, so that $p \notin S$. Consequently

$$\begin{aligned}
\|\chi_r \eta F\|_2^2 &= \int_{\mathbb{R}^n} \chi_r |\eta F|^2 = \sum_p \int_{p+I^n} \chi_r |\eta F|^2 = \sum_p \int_{I^n} \chi_r(x-p)|(\eta F)(x-p)|^2 \, dx \\
&\leq \sum_{p \notin S} \int_{I^n} |\eta(x-p)|^2 |F(x)|^2 \, dx = \int_{I^n} \sum_{p \notin S} |\eta(x-p)|^2 |F(x)|^2 \, dx \leq \varepsilon^2 \|F\|_2^2.
\end{aligned}$$

□

We keep the notation of the Lemma. Since D is unitary, it follows from the Lemma that $\|D^j(\chi_r \eta F)\|_2 \leq \varepsilon \|F\|_2$. Set $\tilde{\chi}_r = 1 - \chi_r$. Then

$$\begin{aligned} |\langle g, D^j(\eta F)\rangle| &\leq |\langle g, D^j(\chi_r \eta F)\rangle| + |\langle g, D^j(\tilde{\chi}_r \eta F)\rangle| \\ &\leq \varepsilon \|g\|_2 \|F\|_2 + |\langle g, D^j(\tilde{\chi}_r \eta F)\rangle|. \end{aligned}$$

Suppose now that 0 is not in the (closed) support of g. Notice that $(D^j(\tilde{\chi}_r \eta F))(x) = \delta^j((\tilde{\chi}_r \eta F)(B^j x))$. If $\tilde{\chi}_r(B^j x) \neq 0$ then $\|B^j x\| \leq r$, so that $\|x\| \leq \|B^{-j}\| r$. Since A is a dilation, the spectral radius of B is < 1. It follows from the spectral radius formula that $\|B^{-j}\| \to 0$ as $j \to -\infty$. Thus for sufficiently negative j the supports of g and $\tilde{\chi}_r \eta F$ are disjoint, so that

$$|\langle g, D^j(\eta F)\rangle| \leq \varepsilon \|g\|_2 \|F\|_2.$$

Putting all this together with our calculations before the Lemma, we find that for j sufficiently negative so that the above estimates apply for all of our finite number of indices k, we have

$$|\langle g, \xi\rangle| \leq \varepsilon (\sum_k \|g\|_2 \|\eta_k\|_{\mathcal{A}} \|\xi\|_2).$$

Since ε is arbitrary, it follows that $\langle g, \xi\rangle = 0$. Thus we have shown that ξ is orthogonal to all $g \in C_c(\mathbb{R}^n)$ whose support does not contain 0. This is sufficient to conclude that $\xi = 0$, as desired. □

We now show that condition (4) of Definition 3.1 holds under a natural condition which is closely related to the familiar condition on scaling functions that $\hat{\varphi}(0) = 1$.

Proposition 3.4. *Let V be a projective $\mathcal{A}$-submodule of Ξ which satisfies conditions (1), (2) and (3) of Definition 3.1. If there is at least one $\xi \in V$ such that $\xi(0) \neq 0$, then V satisfies condition (4) of Definition 3.1.*

Proof. The direction of our proof is suggested by the proof of theorem 1.7 of chapter 2 of [6]. Assume that V satisfies the conditions of the proposition, and set $\mathcal{V} = \bigcup_{-\infty}^{\infty} V_j$. Set $\mathcal{B} = \bigcup_{-\infty}^{\infty} \mathcal{A}_j$, so that $\mathcal{B}$ is a unital $*$-subalgebra of $C_b(\mathbb{R}^n)$ which separates the points of $\mathbb{R}^n$. Let $\overline{\mathcal{V}}$ denote the closure of $\mathcal{V}$ in Ξ. Notice that $\mathcal{V}$ is a $\mathcal{B}$-module since V_j is an $\mathcal{A}_j$-module for each j.

Lemma 3.5. *The subspace $\overline{\mathcal{V}}$ is closed under pointwise multiplication by $C_b(\mathbb{R}^n)$.*

Proof. Let $\xi \in \overline{\mathcal{V}}$ and $F \in C_b(\mathbb{R}^n)$. By Proposition 1.5 we know that $\xi F \in \Xi$, and so it suffices to show that ξF can be approximated in the norm of Ξ by elements of $\mathcal{V}$. Let $\varepsilon > 0$ be given. Let S be a finite subset of $\mathbb{Z}^n$ such that for any $x \in I^n$ we have $\sum_{p \notin S} |\xi(x-p)|^2 < (\varepsilon/\|F\|_\infty)^2$. Let $\overline{\mathcal{B}}$ denote the closure of $\mathcal{B}$ in $C_b(\mathbb{R}^n)$. Then $\overline{\mathcal{B}} \cong C(Y)$ for some compact space Y, and there is a natural continuous injection of $\mathbb{R}^n$ into Y as a dense subset (not open — we are close to the Bohr compactification). Since $I^n - S$ is a compact subset of $\mathbb{R}^n$, its image is a compact subset of Y, on which the restriction to $I^n - S$ of F can be considered to be a continuous function. Thus by the Tietze extension theorem this function extends to a continuous function on Y, and so we can choose a $G \in \mathcal{B}$ such that $\|G\|_\infty \leq \|F\|_\infty$ and $|F(x) - G(x)| < \varepsilon/\|\xi\|_{\mathcal{A}}$ for every $x \in I^n - S$. Set

$H = F - G$. Then for any $x \in I^n$ we have

$$\begin{aligned}\langle \xi H, \xi H\rangle_{\mathcal{A}}(x) &= \sum_{p\in S} |(\xi H)(x-p)|^2 + \sum_{p\notin S} |(\xi H)(x-p)|^2 \\ &\leq (\varepsilon/\|\xi\|_{\mathcal{A}})^2 \sum_{p\in S} |\xi(x-p)|^2 + 2\|F\|_\infty^2 \sum_{p\notin S} |\xi(x-p)|^2 \\ &\leq \varepsilon^2 + 2\varepsilon^2 = 3\varepsilon^2.\end{aligned}$$

It follows that $\|\xi F - \xi G\|_{\mathcal{A}} < \sqrt{3}\varepsilon$. ☐

We now return to the proof of Proposition 3.4. We show that $C_c(\mathbb{R}^n) \subset \overline{\mathcal{V}}$. The conclusion of the proposition then follows from Proposition 1.5. So let $\eta \in C_c(\mathbb{R}^n)$, and let K denote the support of η. Choose a $\xi \in V$ such that $\xi(0) \neq 0$. Since ξ is continuous, $\xi(x) \neq 0$ for all x in some neighborhood of 0. As seen in the proof of Proposition 3.2, we can choose a sufficiently large j that $\|B^j\|$ is very small, small enough that $\xi \circ (B^j)$ is bounded away from 0 on K . Note that $\xi \circ (B^j) \in V_j$ by definition. Set $g = \eta(\xi \circ B^j)^{-1}$. Then $g \in C_c(\mathbb{R}^n)$. Since $\eta = (\xi \circ B^j)g$, it follows from Lemma 3.5 that $\eta \in \overline{\mathcal{V}}$ as desired. ☐

If $k < j$ then $\mathcal{A}_k \subset \mathcal{A}_j$, and so we can also view V_j as an $\mathcal{A}_k$-module.

Proposition 3.6. *If $k < j$, than V_j is projective as an $\mathcal{A}_k$-module.*

Proof. Let V be any projective $\mathcal{A}_j$-module. Then there is an $\mathcal{A}_j$-module W such that $V \oplus W \cong (\mathcal{A}_j)^m$ for some integer m. If $V_{\mathcal{A}_k}$ denotes V viewed as an $\mathcal{A}_k$-module, and similarly for other $\mathcal{A}_j$-modules, then we have

$$V_{\mathcal{A}_k} \oplus W_{\mathcal{A}_k} \cong ((\mathcal{A}_j)^m)_{\mathcal{A}_k} \cong ((\mathcal{A}_j)_{\mathcal{A}_k})^m.$$

We can thus obtain the desired conclusion if we know that A_j is projective as a right $\mathcal{A}_k$-module. But $\mathcal{A}_j$ is actually free as a right $\mathcal{A}_k$-module, with the proof being essentially proposition 1.1 of [10]. Indeed, if we view matters in the time domain, then any set of coset representatives for $A^{-k}\mathbb{Z}^n$ in $A^{-j}\mathbb{Z}^n$ will provide a module basis. ☐

We remark that in the setting of the above proposition a standard module frame for V_j as $\mathcal{A}_k$-module can be constructed as follows from a standard module frame $\varphi_1, \ldots, \varphi_m$ for V_j as an $\mathcal{A}_j$-module. For ease of notation we assume that $k = 0$, so that $j > 0$. We note first that there is a natural $\mathcal{A}$-valued inner product on $\mathcal{A}_j$. The image of $\mathbb{Z}^n$ in $\mathbb{R}^n/B^{-j}\mathbb{Z}^n$ is a finite group, say Γ, which we can pull back to a set, C, of coset representatives for $B^{-j}\mathbb{Z}^n$ in $\mathbb{Z}^n$. Define a linear map E from $\mathcal{A}_j$ onto $\mathcal{A}$ by $E(f)(x) = \sum_{p\in C} f(x-p)$. (Then E is a multiple of a "conditional expectation" as discussed after Corollary 1.12.) We define our $\mathcal{A}$-valued inner product on $\mathcal{A}_j$ by $\langle f, g\rangle_{\mathcal{A}} = E(\bar{f}g)$. The context must distinguish this inner product from that on Ξ.

Since $\mathcal{A}_j$ is free as an $\mathcal{A}$-module, we can find an orthonormal module basis $b_1, \ldots, b_q$ for $\mathcal{A}_j$ as an $\mathcal{A}$-module. Then $\{\varphi_i b_k : 1 \leq i \leq m, 1 \leq k \leq q\}$ will be a standard module frame for V_j as an $\mathcal{A}$-module. To see this, let $\xi \in V_j$. Then

$$\begin{aligned}\xi &= \sum \varphi_i \langle \varphi_i, \xi\rangle_{\mathcal{A}_j} = \sum \varphi_i (\sum b_k \langle b_k, \langle \varphi_i, \xi\rangle_{\mathcal{A}_j}\rangle_{\mathcal{A}}) = \sum (\varphi_i b_k)\langle b_k, \langle \varphi_i, \xi\rangle_{\mathcal{A}_j}\rangle_{\mathcal{A}} \\ &= \sum (\varphi_i b_k) E(\bar{b}_k \langle \varphi_i, \xi\rangle_{\mathcal{A}_j}) = \sum (\varphi_i b_k) E(\langle \varphi_i b_k, \xi\rangle_{\mathcal{A}_j}) = \sum (\varphi_i b_k)\langle (\varphi_i b_k), \xi\rangle_{\mathcal{A}},\end{aligned}$$

so the reconstruction formula holds, as desired.

Suppose now that $\{V_j\}$ is a projective multiresolution analysis. Then V_1 is a projective $\mathcal{A}_1$-module, which contains $V = V_0$, but V is not an $\mathcal{A}_1$-submodule. However, from Proposition 3.6 we can consider V_1 to be a projective $\mathcal{A}$-module, and then V is a projective $\mathcal{A}$-submodule of V_1. From Proposition 2.3 it follows that V has a unique "orthogonal complement", using the $\mathcal{A}$-valued inner product. Let us denote this complement by W_0. Then W_0 is again a projective $\mathcal{A}$-module, and so it will have a standard module frame. These are our wavelets. One way of obtaining a standard module frame for W_0 is as follows. Choose a standard $\mathcal{A}$-module frame for V, and apply D to it to obtain a standard $\mathcal{A}_1$-module frame for V_1. Then apply Proposition 3.6 and the discussion which follows it to obtain a standard $\mathcal{A}$-module frame, say $\varphi_1, \ldots, \varphi_m$, for V_1. For the chosen standard $\mathcal{A}$-module frame for V use Proposition 2.3 to construct the orthogonal projection P from V_1 onto V. Then $Q = P - I$ is the orthogonal projection of V_1 onto W_0. The discussion before Definition 2.1 shows that $Q\varphi_1, \ldots, Q\varphi_m$ will then form a standard $\mathcal{A}$-module frame for W_0, hence they will be our wavelets.

From Proposition 2.5 we see that the $\psi_k e_q$'s will form an ordinary normalized tight frame for the closure of W_0 in $L^2(\mathbb{R}^n)$. In the same way we can let W_j be the orthogonal complement of V_j in V_{j+1} as $\mathcal{A}_j$-modules. As is usual for multiresolution analyses, we have:

Proposition 3.7. *The operator D will carry W_j onto W_{j+1}, and will carry a standard module frame for W_j as $\mathcal{A}_j$-module to one for W_{j+1} as $\mathcal{A}_{j+1}$-module, with a similar relation for the corresponding normalized tight frames.*

Proof. This follows from several straight-forward calculations using Proposition 1.13 together with the fact that D is a unitary operator on $L^2(\mathbb{R}^n)$. ☐

From Corollary 1.12, condition (4) of Definition 3.1, and Proposition 3.7, we obtain:

Theorem 3.8. *The $\mathcal{A}$-submodules V and W_j for $j \geq 0$ are all mutually orthogonal for the $\mathcal{A}$-valued inner product, and the algebraic sum $V \oplus_{j\geq 0}^{\infty} W_j$ is dense in Ξ.*

Let $\overline{V}$ and $\overline{W}_j$ denote the closures of V and W_j in $L^2(\mathbb{R}^n)$. Since Ξ is dense in $L^2(\mathbb{R}^n)$ it follows from Proposition 1.2 that:

Corollary 3.9. *With notation as above we have the Hilbert-space direct sums*

$$L^2(\mathbb{R}^n) = \overline{V} \oplus_{j\geq 0}^{\infty} \overline{W}_j = \oplus_{j\in\mathbb{Z}} \overline{W}_j.$$

Of course in view of this, the union of a normalized tight frame for $\overline{V}$ together with ones for each of the $\overline{W}_j$'s for $j > 0$ is a normalized tight frame for $L^2(\mathbb{R}^n)$. In the same way any collection of normalized tight frames for each of the $\overline{W}_j$'s as j runs over $\mathbb{Z}$ will be a normalized tight frame for $L^2(\mathbb{R}^n)$. The usual pattern for multi-wavelets would be to obtain the tight frames for each $\overline{W}_j$ by first finding a Hilbert $\mathcal{A}_0$-module basis for W_0 (as well as for V), then applying Proposition 3.7, and then applying the j-versions of Proposition 2.5.

From the discussion of topological direct sums of Hilbert C^*-modules given in Chapter 1 of [8] we find that the topological sum $V \oplus_{j\geq 0}^{\infty} W_j$ within Ξ can be identified with the

set

$$\{v + \{w_j\}_{j\geq 0} : w_j \in W_j \text{ and } \sum_{j\geq 0}\langle w_j, w_j\rangle_{\mathcal{A}} \text{ is convergent in } \mathcal{A}\},$$

which is complete. Thus the topological sum $V \oplus_{j\geq 0}^{\infty} W_j$ is a complete submodule of Ξ which contains V_j for all$j \geq 0$. It follows by condition (4) of Definition 3.1 that as a topological sum we have $V \oplus_{j\geq 0}^{\infty} W_j = \Xi$.

We have a similar result for standard module frames. Here we must view each W_j as an $\mathcal{A}$-module, as permitted by Proposition 3.6. (This is not the usual point of view in wavelet theory.) We use the definition of standard module frames for non-finitely-generated modules given in Definition 2.2 .

Theorem 3.10. *Let V and W_j for $j \geq 0$ be as in the statement of Theorem 3.8. Then the union of a standard $\mathcal{A}$-module frame for V and any collection of standard $\mathcal{A}$-module frames for each of the W_j's for $j \geq 0$, is a standard $\mathcal{A}$-module frame for Ξ.*

Proof. Let $\{\varphi_1, \ldots, \varphi_m\}$ be a standard module frame for V, and let $\{\psi_{j,1}, \cdots, \psi_{j,d_j}\}$ be a standard $\mathcal{A}$-module frame W_j for each $j \geq 0$. According to Definition 2.2 we must show that for every $\xi \in \Xi$ we have

$$\langle \xi, \xi\rangle_{\mathcal{A}} = \sum_{i=1}^{m}\langle \varphi_i, \xi\rangle_{\mathcal{A}}\langle \xi, \varphi_i\rangle_{\mathcal{A}} + \sum_{j=0}^{\infty}\sum_{k=1}^{d_j}\langle \xi, \psi_{j,k}\rangle_{\mathcal{A}}\langle \psi_{j,k}, \xi\rangle_{\mathcal{A}},$$

with the sum on the right-hand side converging in norm in $\mathcal{A}$. Fixing $\xi \in \Xi$, we use Theorem 3.8 to write

$$\xi \;=\; v + \sum_{j=0}^{\infty} w_j,$$

where $v \in V,\ w_j \in W_j$ for $j \geq 0$, and by Chapter 1 of [8],

$$\langle \xi, \xi\rangle_{\mathcal{A}} \;=\; \langle v, v\rangle_{\mathcal{A}} + \sum_{j=0}^{\infty}\langle w_j, w_j\rangle_{\mathcal{A}},$$

where the sum on the right-hand side converges in norm in $\mathcal{A}$. But

$$\langle v, v\rangle_{\mathcal{A}} \;=\; \sum_{i=1}^{m}\langle \varphi_i, v\rangle_{\mathcal{A}}\langle v, \varphi_i\rangle_{\mathcal{A}},$$

while

$$\langle w_j, w_j\rangle_{\mathcal{A}} \;=\; \sum_{k=1}^{d_j}\langle \xi_{j,k}, w_j\rangle_{\mathcal{A}}\langle w_j, \xi_{j,k}\rangle_{\mathcal{A}} \quad \text{for each } j \geq 0,$$

so that

$$\langle \xi, \xi\rangle_{\mathcal{A}} = \sum_{i=1}^{m}\langle v, \varphi_i\rangle_{\mathcal{A}}\langle \varphi_i, v\rangle_{\mathcal{A}} + \sum_{j=0}^{\infty}\sum_{k=1}^{d_j}\langle w_j, \psi_{j,k}\rangle_{\mathcal{A}}\langle \psi_{j,k}, w_j\rangle_{\mathcal{A}},$$

where again the right-hand side converges in norm in $\mathcal{A}$. Finally, we note that by the orthogonality of the modules involved,

$$\langle \varphi_i, v\rangle_{\mathcal{A}} \;=\; \langle \varphi_i, \xi\rangle_{\mathcal{A}}, \qquad \text{and} \qquad \langle \psi_{j,k}, w_j\rangle_{\mathcal{A}} \;=\; \langle \psi_{j,k}, \xi\rangle_{\mathcal{A}},$$

for all $1 \le i \le m$, $j \ge 0$, and $1 \le k \le d_j$. From this we see that the condition of Definition 2.2 is satisfied. □

4. A review of finitely generated projective $C(\mathbb{T}^2)$ modules

In this section, we review results on finitely generated projective modules over $\mathcal{A} = C(\mathbb{T}^2)$ which we will use often in later parts of the paper. These results are module versions of known results about complex vector bundles over $\mathbb{T}^2$ which follow easily from elementary vector bundle considerations, such as those given in [1] [7]. The modules we consider were constructed in Section 3 of [14] to study the rational rotation algebras $\mathcal{A}_{p/q}$ with phase factor $e(p/q)$. The approach here will be slightly different from that in [14]. Throughout this section, unless otherwise specified, we again view functions on $\mathbb{T}^n$ as functions defined on $\mathbb{R}^n$ which are periodic modulo $\mathbb{Z}^n$. The following proposition summarizes the results from theorem 3.9 of [14].

Proposition 4.1. *For q, $a \in \mathbb{Z}$ with $q \neq 0$ let $X(q,a)$ denote the right $\mathcal{A}$-module consisting of the space of continuous complex-valued functions F on $\mathbb{T} \times \mathbb{R}$ which satisfy*

$$F(s, t-q) \;=\; e(as)F(s,t),$$

with right $\mathcal{A}$-module action given by

$$(Ff)(s,t) \;=\; F(s,t)f(s,t),$$

for $F \in X(q,a)$ and $f \in \mathcal{A}$. Then $X(q,a)$ is a finitely generated, projective $\mathcal{A}$-module. The set $\{X(q,a) :\; q,\, a \in \mathbb{Z},\; q > 0\}$ parametrizes the isomorphism classes of finitely generated projective $\mathcal{A}$-modules, in the sense that if X is a finitely generated projective $\mathcal{A}$-module, then there exist unique values of q and a such that $X \cong X(q,a)$. For $q > 0$ we say that $X(q,a)$ has dimension q and twist $-a$.

We define an $\mathcal{A}$-valued inner product on $X(q,a)$ by

$$\langle F,\, G\rangle_{\mathcal{A}} \;=\; \sum_{0 \le k \le q-1} \overline{F(s,t-k)}G(s,t-k),$$

for $F, G \in X(q,a)$. The module $X(q,a)$ is complete for the norm determined by this inner product, so that it is a Hilbert $\mathcal{A}$-module.

A simple calculation shows that $X(-q,a) \cong X(q,-a)$. We note that there is an asymmetry in the treatment of the variables in the formula defining the functions in $X(q,a)$. This is just a matter of convention. But it will affect later formulas which we use.

We now give a more detailed description of the $X(q,a)$'s. First we fix some notation which will also be useful later. For q, $a \in \mathbb{Z}$ with $q > 0$, and for $\beta \in \mathbb{R}$ define the function $J_{q,a,\beta} : \mathbb{R}^2 \to \mathbb{C}$ by

$$J_{q,a,\beta}(s,t) \;=\; e(nas) \quad \text{for} \quad t \in [\beta + nq, (\beta + nq) + q), \quad n \in \mathbb{Z}.$$

We note that since $\mathbb{R}$ is the disjoint union $\cup_{n\in\mathbb{Z}}[\beta + nq, (\beta + nq) + q)$, the above formula defines $J_{q,a,\beta}(s,t)$ on all of $\mathbb{R}^2$. By construction,

$$J_{q,a,\beta}(s,t-q) \;=\; e(as)J_{q,a,\beta}(s,t);$$

but $J_{q,a,\beta}(s,t)$ is not an element of $X(q,a)$ because it is not continuous. However, if $f : \mathbb{R} \to \mathbb{C}$ is any q-periodic continuous function such that $f(\beta) = 0$ (so that $f(\beta+nq) = 0$

for all $n \in \mathbb{Z}$ by the q-periodicity of f), then it is easy to verify that the function $J_{q,a,\beta}(s,t)f(t)$ will be an element of $X(q,a)$. In fact, many infinitely differentiable functions can be expressed in this way.

It is possible to show that if we fix β_0 with $0 < \beta_0 < q$, then the functions of the form $J_{q,a,\beta}(s,t)f(t)$ for $\beta = 0$ together with those for $\beta = \beta_0$, and for f continuous, q-periodic and with $f(\beta) = 0$, form a set of $C(\mathbb{T}^2)$-generators for $X(q,a)$. But as we do not need such a result here, we do not include a proof. The functions $J_{q,a,\beta}$ together with low-pass filter functions can be used to construct standard module frames for each $X(q,a)$, and thus also the corresponding embeddings into free modules. We give the procedure for $q = 1$ and leave the details for the general case to the reader, since we will not need them later. So let $q = 1$ and let $a \in \mathbb{Z}$ be arbitrary. Let m_0 be a continuous low-pass filter defined on $\mathbb{R}$ for dilation by 2, so that, aside from a constant such as $\sqrt{2}$, we have

$$\begin{aligned} m_0(0) &= 1, \text{ and} \\ |m_0(t)|^2 + |m_0(t+1/2)|^2 &= 1, \quad \text{so that } m_0(1/2) = 0. \end{aligned}$$

Set $h_1(s,t) = m_0(t)J_{1,a,0}(s,t)$ and $h_2(s,t) = m_0(t+1/2)J_{1,\,a,\,1/2}(s,t)$. Then $h_1,\ h_2 \in X(q,a)$, and it is easily verified that they form a standard module frame for $X(1,\ a)$. We refer the reader to Proposition 2.1 of [9] and to [14] for ideas on how to go about extending this procedure for general q.

Theorem 4.2. *(c.f.* [14]*, Theorem 3.9) Let $q_1,\ q_2,\ a_1,\ a_2 \in \mathbb{Z}$ with $q_1 > 0,\ q_2 > 0$ be given. Then*

$$X(q_1,a_1) \oplus X(q_2,a_2) \cong X(q_1+q_2, a_1+a_2)$$

as finitely generated projective $\mathcal{A}$-modules. In particular, cancellation holds for all finitely generated $\mathcal{A}$-modules; that is, if

$$X(q_1,a_1) \oplus X(q_2,a_2) \cong X(q_1,a_1) \oplus X(q_3,a_3),$$

then $X(q_2,a_2) \cong X(q_3,a_3)$, so that $q_2 = q_3$ and $a_2 = a_3$.

Since it is evident that $X(1,0)$ is isomorphic to $\mathcal{A}$ viewed as a right $\mathcal{A}$-module, it follows from the above theorem that $X(q,0)$ is isomorphic to the free $\mathcal{A}$-module $\mathcal{A}^q$. Our previous paper [10] can be used to prove in a different manner that $X(q,0) \cong \mathcal{A}^q$, and as mentioned there, filter functions for dilation of $\mathbb{R}$ by q can be used to construct module bases for $X(q,0)$. We remark that, as discussed in [14], $K_0(\mathcal{A}) \cong \mathbb{Z}^2 = \{(q,a) :\ q,a \in \mathbb{Z}\}$, consistent with the above parametrization, so that the positive cone of $K_0(\mathcal{A})$ is equal to $\mathbb{N} \times \mathbb{Z} \cup \{(0,0)\} \subseteq \mathbb{Z}^2$. Also, cancellation fails for complex vector bundles over tori of dimension ≥ 5, and this phenomenon played a crucial role in our discussion of the (non)-existence of wavelet filter functions in [10].

Now Theorem 4.2 shows that there is a Hilbert $\mathcal{A}$-module isomorphism between $X(q,a)$ and $X(1,a) \oplus \mathcal{A}^{q-1}$. From the above observations we then obtain:

Proposition 4.3. *Let $q \in \mathbb{N}$, and $a \in \mathbb{Z}\setminus\{0\}$. Then there is a module frame in $X(q,a)$ which consists of $q+1$ elements.*

It is possible to explicitly write down this module frame in terms of wavelet filter functions. We leave this as an exercise for the interested reader.

5. The "scaling function"

In this section we restrict attention to the case of dimension 2, and to diagonal dilation matrices. For this case we show how to produce many non-free projective $\mathcal{A}$-submodules V of Ξ which produce projective multiresolution analyses as defined in Definition 3.1. Throughout this section we assume that $\mathcal{A} = C(\mathbb{T}^2)$, and we fix our dilation matrix A to be of the form $A = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ for $d_i \in \mathbb{Z}$ with $|d_i| > 1$ for $i = 1, 2$. Thus our earlier δ will be $\delta = (|d_1 d_2|)^{-1/2}$. Also, Ξ will be a subspace of $L^2(\mathbb{R}^2)$.

In the next theorem the function σ acts much like a traditional scaling function, with condition 2 containing the analog of the traditional scaling equation.

Theorem 5.1. *Fix integers $q, a \in \mathbb{Z}$ with $q > 0$. Let $\sigma \in \Xi$ satisfy the further conditions that:*

(1) $\langle \sigma, \sigma \rangle_{C(\mathbb{R}^2/(\mathbb{Z}\times q\mathbb{Z}))} = 1$, *where we use the notation analogous to that of Section 1.*
(2) *There is an $\tilde{m} \in X(q, (1 - d_1 d_2)a)$ such that*

$$\sigma(Ax) = \tilde{m}(x)\sigma(x)$$

for all $x \in \mathbb{R}^2$.

Define $\mathcal{R} : X(q, a) \to \Xi$ by the pointwise product

$$\mathcal{R}(F) = \sigma F.$$

Then $\mathcal{R}$ is an $\mathcal{A}$-module monomorphism which has the further properties that:

(1) $\langle \mathcal{R}(F), \mathcal{R}(G)\rangle_{\mathcal{A}} = \langle F, G\rangle_{\mathcal{A}}$, *for all $F, G \in X(q, a)$,*
(2) $\mathcal{R}(X(q, a)) \subseteq D\mathcal{R}(X(q, a))$

In particular, $\mathcal{R}(X(q, a))$ is a projective $\mathcal{A}$-submodule of Ξ.

Proof. A straight-forward calculation, using coset representatives for $q\mathbb{Z}$ in $\mathbb{Z}$, verifies property (1) above. In particular, this shows that $\mathcal{R}(F)$ is indeed in Ξ. Since the module action is pointwise multiplication on both sides, it is clear that $\mathcal{R}$ is an $\mathcal{A}$-module homomorphism. From property (1) it is also clear that $\mathcal{R}$ is a monomorphism.

To verify property (2) we must show that if $F \in X(q, a)$ then $D^{-1}\mathcal{R}F \in \mathcal{R}X(q, a)$. Recall that $(D\xi)(x) = \delta\xi((A^t)^{-1}x)$. For our diagonal matrix A we have $A^t = A$, and so $(D^{-1}\xi)(x) = \delta^{-1}\xi(Ax)$. Thus we must show that $(\sigma F) \circ A = \sigma G$ for some $G \in X(q, a)$. Now

$$((\sigma F) \circ A)(x) = \sigma(Ax)F(Ax) = \sigma(x)\tilde{m}(x)F(Ax),$$

and so we define G by $G(x) = \tilde{m}(s)F(Ax)$. Then

$$\begin{aligned} G(s+1, t+q) &= \tilde{m}(s+1, t+q)F(d_1(s+1), d_2(t+q)) \\ &= e((1-d_1d_2)as)\tilde{m}(s,t)e(d_2ad_1s)F(d_1s, d_2t) = e(as)G(s,t). \end{aligned}$$

Thus $G \in X(q, a)$ as desired. $\square$

Our task now is to show the existence of functions σ and $\tilde{m}$ satisfying the conditions of the above theorem. These conditions impose further conditions on $\tilde{m}$.

Proposition 5.2. *Let σ and $\tilde{m}$ satisfy the conditions of the above theorem. Let C be a set of coset representatives for $A(\mathbb{Z}\times q\mathbb{Z})$ in $\mathbb{Z}\times q\mathbb{Z}$. Then*

$$\sum_{c\in C}|\tilde{m}(x-A^{-1}c))|^2=1.$$

In other words, $\langle \tilde{m},\tilde{m}\rangle_{C(\mathbb{R}^2/A^{-1}(\mathbb{Z}\times q\mathbb{Z}))}=1.$

Proof. Using conditions (1) and (2) of the above theorem, we calculate, for any $x\in\ \mathbb{R}^2$:

$$\begin{aligned}
1 &= \langle\sigma,\ \sigma\rangle_{C(\mathbb{Z}\times q\mathbb{Z})}(Ax)=\sum_{p\in\mathbb{Z}\times q\mathbb{Z}}|\sigma(Ax-p)|^2=\sum_{c\in C}\sum_{p\in\mathbb{Z}\times q\mathbb{Z}}|\sigma(Ax-c-Ap|^2\\
&= \sum_c\sum_p|\sigma(A(x-A^{-1}c-p))|^2=\sum_c\sum_p|\tilde{m}(x-A^{-1}c-p)|^2|\sigma(x-A^{-1}c-p)|^2\\
&= \sum_c|\tilde{m}(x-A^{-1}c)|^2\langle\sigma,\ \sigma\rangle_{C(\mathbb{Z}\times q\mathbb{Z})}(x-a^{-1}c)=\sum_c|\tilde{m}(x-A^{-1}c)|^2
\end{aligned}$$

□

If as coset representatives we choose the integer pairs (j,k) such that $0\le j\le |d_1|-1$ and $0\le k\le |d_2|-1$, then the above equation can be written as

$$\sum_{0\le j\le |d_1|-1}\ \sum_{0\le k\le |d_2|-1}|\tilde{m}(s+\frac{j}{d_1},t+\frac{kq}{d_2})|^2=1$$

for all $s,t,\in\ \mathbb{R}$. When $q=1$, this equation is closely related to one of the standard equations that a low-pass filter must satisfy in ordinary wavelet theory corresponding to dilation by the matrix A, except that now $\tilde{m}\ \in\ X(q,(1-d_1d_2)a)$.

Following the pattern for ordinary scaling functions, we seek to define σ by

$$\sigma(s,t)\ =\ \prod_{i=1}^{\infty}\tilde{m}(A^{-i}(s,t)).$$

This can be done directly, but we choose to use facts from ordinary wavelet theory to by-pass having to deal directly with this infinite product. But our discussion below is equivalent to finding conditions on $\tilde{m}$ which assure that this product converges to a function having the properties which we need.

As usual, we will insist that $\tilde{m}(0,0)=1$. Then, as in ordinary wavelet theory, from the equation in the above proposition it follows that $\tilde{m}(0,kq/d_2)=0$ for $1\le k\le |d_2|-1$. We will seek $\tilde{m}$ of the form $\tilde{m}(s,t)\ =\ J_{q,(1-d_1d_2)a,\beta}\ m$ where m is an ordinary low-pass filter function for dilation by A, but for translation by $\mathbb{Z}\times q\mathbb{Z}$ instead of $\mathbb{Z}^2$ (so that $m\in X(q,0)$). But even more, we take m to be a tensor product of ordinary one-dimensional low-pass filters, m_1 and m_2, where m_1 will be for dilation by d_1, while m_2 will be for dilation by d_2 but also for translation by $q\mathbb{Z}$ instead of $\mathbb{Z}$. (Again, we normalize our low-pass filters to have value 1 at 0.) According to theorem 3.2 of [3], and especially its claim 3.3, we can choose m_1 and m_2 to be smooth, and to give corresponding scaling functions φ_1 and φ_2 which are Schwartz functions.

We will define φ by $\varphi(s,t)\ =\ \varphi_1(s)\varphi_2(t)$ for $s,\ t\ \in\ \mathbb{R}$. Then φ itself is a Schwartz function, and so is in Ξ according to Corollary 1.7. Clearly φ is the ordinary scaling

function for m, and $\langle\varphi,\ \varphi\rangle_{C(\mathbb{R}^2/(\mathbb{Z}\times q\mathbb{Z}))} = 1$. We will have $m_2(0) = 1$, so that $m_2(q/d_2) = 0$ as usual, and thus we will have $m(s, q/d_2) = 0$ for all s. Thus we will take $\beta = q/d_2$ so that $\tilde{m} = J_{q,(1-d_1d_2)a,q/d_2} m$ will be a continuous function.

From the infinite product indicated above for σ it is easy to guess that σ will be of the form $\sigma = J\varphi$ for a suitable function J. We can expect that J will not be continuous, but it should be bounded and have its discontinuities where φ takes value 0. Now as usual φ satisfies the scaling equation

$$\varphi(Ax) = m(x)\varphi(x),$$

that is,

$$\varphi(d_1s, d_2t) = m(s,t)\varphi(s,t).$$

Since $m(s, (q/d_2) + nq) = 0$ for any $n \in \mathbb{Z}$, we find, upon iterating, that φ will take value 0 on the vertical lines with $t = (1 + d_2n)d_2^{j-1}q$ for $j \geq 1$ and $n \in \mathbb{Z}$. Notice that all of these values of t are integers not equal to 0. For simplicity of exposition we will ignore in the next proposition what happens on these vertical lines.

Proposition 5.3. *Define on $\mathbb{R}^2$ a function, J, by*

$$J(x) = \prod_{j=1}^{\infty} J_{q,(1-d_1d_2)a,q/d_2}(A^{-j}(x)).$$

This product converges uniformly on any compact subset of $\mathbb{R}^2$ which does not meet the vertical lines $(s, (1+d_2n)d_2^{j-1}q)$. Thus it is a continuous function except possibly on those vertical lines. Furthermore, $|J(x)| = 1$ except possibly on those vertical lines. Finally,

$$J(Ax) = J_{q,(1-d_1d_2)a,q/d_2}(x)J(x)$$

for x not on those vertical lines.

Proof. For any $j \geq 1$ we have

$$J_{q,(1-d_1d_2)a,q/d_2}(A^{-j}(s,t)) = J_{q,(1-d_1d_2)a,q/d_2}(d_1^{-j}s, d_2^{-j}t)$$

for all $x = (s,t)$. So the discontinuities of this function occur exactly where $d_2^{-j}t = (q/d_2)+nq$, which gives exactly the vertical lines found above. The function $J_{q,(1-d_1d_2)a,q/d_2}$ is infinitely differentiable at $(0,0)$ and has value 1 there, and so the usual proof for the construction of scaling functions works here too, when applied to points not on the vertical lines. See, for example, section 2.3 of [6]. A simple standard argument then verifies the above scaling equation for J. Further examination using the fact that $|J_{q,(1-d_1d_2)a,q/d_2}| = 1$ shows that $|J| = 1$. $\square$

Theorem 5.4. *With notation as above, set $\sigma = J\varphi$. Then σ satisfies the conditions of Theorem 5.1.*

Proof. Because φ is continuous, and because J is bounded and its discontinuities occur exactly where φ takes value 0, it follows that σ is continuous. Furthermore, for any $x \in \mathbb{R}^2$ we have

$$\sigma(Ax) = J(Ax)\varphi(Ax) = J_{q,(1-d_1d_2)a,q/d_2}(x)J(x)m(x)\varphi(x) = \tilde{m}(x)\sigma(x).$$

Finally, since $|J_{q,(1-d_1d_2)a,q/d_2}| = 1$, we have $\langle\sigma,\sigma\rangle_{\mathcal{A}} = \langle\varphi,\varphi\rangle_{\mathcal{A}}$ so that $\sigma \in \Xi$. $\square$

Theorem 5.5. *For our given dilation matrix A and for a given $X(q,a)$ there are projective multiresolution analyses $\{V_j\}$ such that $V_0 \cong X(q,a)$.*

Proof. We have obtained a function σ satisfying the conditions of Theorem 5.1, from which we obtain the corresponding module monomorphism $\mathcal{R}$ of $X(q,a)$ into Ξ. We set $V = V_0 = \mathcal{R}(X(q,a)$. From Theorem 5.1 we see that $V \subseteq D(V)$. For each $j \in \mathbb{Z}$ we set $V_j = D(V)$. Then from Propositions 3.2 and 3.4 we see that the family $\{V_j\}$ of subspaces of Ξ is a projective multiresolution analysis as defined in Definition 3.1. $\square$

6. The structure of the wavelet module

From the construction of the projective multiresolution analyses just given we can then construct the corresponding wavelet modules W_j as discussed before Theorem 3.8. We now come another of our main results, which is the identification of the isomorphism class of W_0 as projective $\mathcal{A}$-module. In order to do this we must first identify the isomorphism class of V_1 as $\mathcal{A}$-module.

Recall that for our given "scaling function" σ we have

$$V_1 = \{(\sigma F) \circ B : F \in X(q,a)\}.$$

Set $Y_A(q,a) = \{F \circ B : \ F \in X(q,a)\}$, with its evident structure as an $\mathcal{A}$-module by pointwise multiplication. The mapping $(F \circ B) \mapsto (\sigma \circ B)(F \circ B)$ is clearly an $\mathcal{A}$-module isomorphism from $Y_A(q,a)$ onto V_1. (So $Y_A(q,a)$ must be projective.) Thus it suffices for us to determine the isomorphism class of $Y_A(q,a)$.

Now $(F \circ B)(s,t) = F(s/d_1, t/d_2)$, and simple calculations show that equivalently we have

$$Y_A(q,a) = \{G \in C_b(\mathbb{R}^2) : G(s-d_1,t) = G(s,t) \quad \text{and} \quad G(s,t-d_2q) = e((a/d_1)s)G(s,t)\}.$$

Notice that a/d_1 need not be an integer. To deal with this we let c be the greatest common divisor of a and d_1, chosen so that $c > 0$, and we define $\hat{a}$ and $\hat{d}_1$ by $a = \hat{a}c$ and $d_1 = \hat{d}_1 c$.

For any $p,q,a \in \mathbb{Z}$ with $p \neq 0$ and $q \neq 0$ set

$$Z(p;q,a) = \{F \in C_b(\mathbb{R}^2) : F(s-p,t) = F(s,t) \quad \text{and} \quad F(s,t-q) = e(as)F(s,t)\}.$$

Lemma 6.1. *The $\mathcal{A}$-module $Z(d_1;\hat{d}_1 d_2 q, \hat{a})$ is isomorphic to the direct sum of $|\hat{d}_1|$ copies of $Y_A(q,a)$, which we denote by $|\hat{d}_1| Y_A(q,a)$.*

Proof. Define an automorphism, α, of $Z(d_1;\hat{d}_1 d_2 q, \hat{a})$ by

$$(\alpha F)(s,t) = \bar{e}((\hat{a}/\hat{d}_1)s)F(s,t-d_2q).$$

It is easily checked that this is indeed a module automorphism, and that it is of order $|\hat{d}_1|$. That is, the cyclic group of order $|\hat{d}_1|$ acts on the module. Then $Z(d_1;\hat{d}_1 d_2 q, \hat{a})$ decomposes into the direct sum of its "isotypic components" for this action. The fixed-point submodule clearly consists of the F's in $Z(d_1;\hat{d}_1 d_2 q, \hat{a})$ satisfying

$$F(s,t-d_2q) = e((a/d_1)s)F(s,t)\}.$$

That is, the fixed-point submodule actually is $Y_A(q,a)$. For any $k \in \mathbb{Z}$ with $0 \leq k \leq |\hat{d}_1|-1$ the k-th isotypic component consists of the F's which satisfy $\alpha(F) = e(k/\hat{d}_1)F$. But for such an F set $F_0(s,t) = e(kt/(\hat{d}_1 d_2 q))F(s,t)$. Then it is easily checked that $F_0 \in$

$Y_A(q,a)$, and that this gives an $\mathcal{A}$-module isomorphism of the k-th isotypic component onto $Y_A(q,a)$. $\square$

So we see that we must now determine the isomorphism class of modules of the form $Z(p;q,a)$.

Proposition 6.2. *For and $p,q,a \in \mathbb{Z}$ with $p \neq 0$ and $q \neq 0$ the $\mathcal{A}$-module $Z(p;q,a)$ is isomorphic to the direct sum of $|p|$ copies of $X(q,a)$, and so to $X(pq,pa)$.*

Proof. Define an automorphism, α, of $Z(p;q,a)$ by $(\alpha F)(s,t) = F(s-1,t)$. Then α has order $|p|$, so that the cyclic group of order $|p|$ acts, and $Z(p;q,a)$ decomposes into the direct sum of its isotypic components. The fixed-point submodule is clearly $X(q,a)$. For any $k \in \mathbb{Z}$ with $0 \leq k \leq |p|-1$ the k-th isotypic component consists of the F's which satisfy $\alpha(F) = e(k/p)F$. But for such an F set $F_0(s,t) = \bar{e}(ks/p)F(s,t)$. It is easily checked that $F_0 \in Y(q,a)$, and that this gives a module isomorphism of the k-th isotypic component onto $Y(q,a)$. Thus $Z(p;q,a)$ is isomorphic to $|p|$ copies of $Y(q,a)$, which we can denote by $|p|Y(q,a)$. But by Theorem 4.2 this in turn is isomorphic to $X(pq,pa)$. $\square$

Theorem 6.3. *With notation as above, as $\mathcal{A}$-modules we have*

$$V_1 \cong Y_A(q,a) \cong X((\det(A))q,a).$$

Proof. We saw above that $V_1 \cong Y_A(q,a)$. Upon applying successively Proposition 6.1, Proposition 6.2, and Theorem 4.2, we find that

$$|\hat{d}_1|Y_A(q,a) \cong Z(d_1;\hat{d}_1d_2q,\hat{a}) \cong |d_1|X(\hat{d}_1d_2q,\hat{a}) \cong |\hat{d}_1|X(d_1d_2q,c\hat{a}).$$

We know that $Y_A(q,a)$ is a projective module, and so by the cancellation property stated as part of Theorem 4.2 we obtain the desired conclusion. $\square$

Theorem 6.4. *Let W_0 be the wavelet space for the projective multiresolution analysis based as above on the projective module V isomorphic to $X(q,a)$, for the diagonal dilation matrix A. Then*

$$W_0 \cong X\big((|\det(A)|-1)q, (\operatorname{sign}(\det(A))-1)a\big).$$

Thus if $\det(A) > 0$ then W_0 is a free module, while if $\det(A) < 0$ then W_0 is not a free module as long as $a \neq 0$.

Proof. From the comment before Theorem 4.2 we have

$$X((\det(A))q,a) = X(|\det(A)|q, \operatorname{sign}(\det(A))a).$$

From the above theorem we see that

$$W_0 \oplus X(q,a) \cong X\big(|\det(A)|q, \operatorname{sign}(\det(A))a\big).$$

The desired conclusion then follows from Theorem 4.2. $\square$

Both Theorem 5.5 and Theorem 6.4 deal with the case where the dilation matrix is a 2×2 diagonal matrix, and it is natural to ask whether or not it is possible to build a projective multiresolution analysis whose initial module V is isomorphic to any $C(\mathbb{T}^2)$-module for any 2×2 integer dilation matrix. As mentioned before, Bownik and Speegle have shown that for any such dilation matrix, there exist scaling functions whose Fourier transforms are continuous and compactly supported [3]. For this reason, we conjecture

that such a construction is possible. However, to date, we have only been able to perform this construction for matrices that are similar to diagonal matrices, and for the dilation matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ corresponding to the quincunx lattice.

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